EFFICINCY

Definition:

Let θ_1^* and θ_2^* are two unbiased estimator if the variance of $\theta_1^* < \theta_2^*$ then θ_1^* is called the efficient estimator as compare to θ_2^*

Then relative efficiency= $\frac{V(\theta_1^*)}{V(\theta_2^*)} \times 100$

Procedure:

Compute the variance of the unbiaesd estimator, which estimator having minimum variance is called efficient one as compare to other.

a) When estimator are unbiased:

Let θ_1^* and θ_2^* are unbiased estimator of parameter θ .Then θ_1^* is called efficient estimator of θ if $V(\theta_1^*) < V(\theta_2^*)$

Relative efficiency= $\frac{V(\theta_1^*)}{V(\theta_2^*)} \times 100$

b) When estimator is biased:

Or

Biased estimator

In this situation we find the mean square error of these estimators.

$MSE(\theta_1^*) = Var(\theta_1^*) + (bias(\theta_1^*))^2$

For 2nd

$MSE(\theta_2^*) = Var(\theta_2^*) + (bias(\theta_2^*))^2$

MSE which is less then other that is efficient estimator.

Relative efficiency= $\frac{MSE(\theta_1^*)}{MSE(\theta_2^*)}$

Q.1: Show that an estimating average random sampling from a normal population. Sample mean is more efficient than sample median. Also find their efficiency?

SOLUTION:

As we know

$\text{var}(\overline{X}) = \delta^2 / n$ $\qquad\qquad \text{var}(X) = \delta^2$

$\text{var}(\overline{X}) = (samplemean) = \delta^2 / n$

And

$\text{var}(\tilde{X}) = (samplemedian) = \pi\delta^2 / 2n$

$$\mathrm{var}(\tilde{X}) = \frac{\frac{22}{7}\delta^2}{2n}$$

$$\mathrm{var}(\tilde{X}) = 3.1415\,\delta^2/2n$$

So

$$\mathrm{var}(\tilde{X}) \geq \mathrm{var}(\overline{X})$$

So sample mean is more efficient than sample median

As

$$R.E = \mathrm{var}(\overline{X})/\mathrm{var}(\tilde{X}) \times 100$$

$$R.E = \frac{\delta^2/n}{1.57\delta^2/n} \times 100$$

R.E=63.69%

As sample mean is 63.69% more efficient as compare to sample median

Or

100-63.69=36.69%

Sample median is 36.69% less efficient than sample mean

Q.2: If 'T' is most efficient and "T'"is less efficient estimator with efficiency "E" and if the correlation coefficient between T and "T'" is "ρ "considering the estimator "T" define by

$$(1+E-2\rho\sqrt{E})T'' = (1-\rho\sqrt{E})T + (E-\rho\sqrt{E})T'$$

Show that

$$\rho = \sqrt{E}$$

SOLUTION:

Let

$$(1+E-2\rho\sqrt{E})T'' = (1-\rho\sqrt{E})T + (E-\rho\sqrt{E})T'$$

Apply variance on both sieds

$$V[(1+E-2\rho\sqrt{E})T''] = V[(1-\rho\sqrt{E})T + (E-\rho\sqrt{E})T']$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = (1-\rho\sqrt{E})^2 V(T) + (E-\rho\sqrt{E})^2 V(T') + 2COV[(1-\rho\sqrt{E})T, (E-\rho\sqrt{E})T']$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = (1-\rho\sqrt{E})^2 V(T) + (E-\rho\sqrt{E})^2 V(T') + 2(1-\rho\sqrt{E}),(E-\rho\sqrt{E})COV(T,T')$$

Let

$$V(T) = V$$

As we know that

$$E = {}^{V(T)}\!/_{V(T')}$$

$$E = {}^{V}\!/_{V(T')}$$

$$V(T') = {}^{V}\!/_{E}$$

AS we know that correlation coefficient

$$\rho = \frac{COV(T,T')}{\sqrt{Var(T),Var(T')}}$$

$$\rho = \frac{COV(T,T')}{\sqrt{V.\frac{V}{E}}}$$

$$COV(T,T') = \rho \frac{V}{\sqrt{E}}$$

Now put

$$(1+E-2\rho\sqrt{E})^2 V(T'') = (1-\rho\sqrt{E})^2 V + (E-\rho\sqrt{E})^2 \frac{V}{E} + 2(1-\rho\sqrt{E})(E-\rho\sqrt{E})\rho\frac{V}{\sqrt{E}}$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[(1+\rho^2 E-2\rho\sqrt{E}) + (E^2+\rho^2 E-2\rho E\sqrt{E})\frac{1}{E} + 2(E-\rho\sqrt{E}-\rho E\sqrt{E}+\rho^2 E)\frac{\rho}{\sqrt{E}}]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[(1+\rho^2 E-2\rho\sqrt{E}) + (E^2+\rho^2 E-2\rho E\sqrt{E})\frac{1}{E} + (2E-2\rho\sqrt{E}-2\rho E\sqrt{E}+2\rho^2 E)\frac{\rho}{\sqrt{E}}]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[1+\rho^2 E-2\rho\sqrt{E}+E+\rho^2-2\rho\sqrt{E}+2\rho\sqrt{E}-2\rho^2-2\rho^2 E+2\rho^3\sqrt{E}]$$

Therefore $E=\sqrt{E}\sqrt{E}$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[1-\rho^2 E-2\rho\sqrt{E}-\rho^2+E+2\rho^3\sqrt{E}]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[1+E-2\rho\sqrt{E}-\rho^2 E-\rho^2+2\rho^3\sqrt{E}]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[1(1+E-2\rho\sqrt{E})-\rho^2(1+E-2\rho\sqrt{E})]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[(1-\rho^2)(1+E-2\rho\sqrt{E})]$$

$$(1+E-2\rho\sqrt{E})^2 V(T'') = V[(1-\rho^2)(1+E-2\rho\sqrt{E})]$$

$$V(T'') = {}^{V(1-\rho^2)}\!/_{(1+E-2\rho\sqrt{E})}$$

It is given that ‘T’ is more efficient

$$V(T) \le V(T'')$$

$$V \le {}^{V(1-\rho^2)}\!/_{(1+E-2\rho\sqrt{E})}$$

$$(1+E-2\rho\sqrt{E}) \le (1-\rho^2)$$

$$(1+E-2\rho\sqrt{E}-1+\rho^2) \le 0$$

$(E - 2\rho\sqrt{E} + \rho^2) \leq 0$

$(\sqrt{E} - \rho)^2 \leq 0$

It is always positive so

$(\sqrt{E} - \rho)^2 = 0$

Taking square root on both sides

$\sqrt{E} - \rho = 0$

$\sqrt{E} = \rho$

As required result

Q.3: If E_1 and E_2 are two efficient of unbiased estimators t_1 and t_2 and ρ is the correlation coefficient between t_1 and t_2 then show that

$$\sqrt{E_1E_2} - \sqrt{(1-E_1)(1-E_2)} \leq \rho \leq \sqrt{E_1E_2} + \sqrt{(1-E_1)(1-E_2)}$$

Solution:

Let

$$E_1 = \frac{V}{V(T_1)} \,\&\, E_2 = \frac{V}{V(T_2)}$$

And suppose 'v' be the minimum variance

Then $V(T_1) = \frac{V}{E_1}$ and $V(T_2) = \frac{V}{E_2}$

Now consider another estimator which is also unbiased.

$T = \lambda_1T_1 + \lambda_2T_2$ $\qquad \therefore \lambda_1 + \lambda_2 = 1$

Applying variance on both sides

$$V(T) = V(\lambda_1T_1) + V(\lambda_2T_2) + 2COV(\lambda_1T_1, \lambda_2T_2)$$

$$V(T) = \lambda_1{}^2V(T_1) + \lambda_2{}^2V(T_2) + 2\lambda_1\lambda_2COV(T_1, T_2)$$

As we know that

$$\rho = \frac{\operatorname{cov}(T_1, T_2)}{\sqrt{V(T_1), V(T_2)}}$$

$$\rho = \frac{\operatorname{cov}(T_1, T_2)}{\sqrt{\frac{V}{E_1}, \frac{V}{E_2}}}$$

$$\operatorname{cov}(T_1, T_2) = \rho\frac{V}{\sqrt{E_1, E_2}}$$

Now replacing values

$$V(T) = \lambda_1{}^2 \frac{V}{E_1} + \lambda_2{}^2 \frac{V}{E_2} + 2\lambda_1\lambda_2\rho \frac{V}{\sqrt{E_1, E_2}}$$

$$V(T) = V[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}]$$

As we know variance (v) is minimum

$$V(T) \geq V$$

$$V[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}] \geq V$$

$$[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}] \geq 1$$

$$[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}] \geq (1)^2$$

$$[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}] \geq (\lambda_1 + \lambda_2)^2$$

$$[\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}}] \geq (\lambda_1{}^2 + \lambda_2{}^2 + 2\lambda_1\lambda_2)$$

$$\frac{\lambda_1{}^2}{E_1} + \frac{\lambda_2{}^2}{E_2} + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}} - \lambda_1{}^2 - \lambda_2{}^2 - 2\lambda_1\lambda_2 \geq 0$$

$$\frac{\lambda_1{}^2}{E_1} - \lambda_1{}^2 + \frac{\lambda_2{}^2}{E_2} - \lambda_2{}^2 + \frac{2\lambda_1\lambda_2\rho}{\sqrt{E_1, E_2}} - 2\lambda_1\lambda_2 \geq 0$$

$$\lambda_1{}^2(\frac{1}{E_1} - 1) + \lambda_2{}^2(\frac{1}{E_2} - 1) + 2\lambda_2(\frac{\rho}{\sqrt{E_1, E_2}} - 1)\lambda_1 \geq 0$$

$$\lambda_1{}^2(\frac{1}{E_1} - 1) + 2\lambda_2(\frac{\rho}{\sqrt{E_1, E_2}} - 1)\lambda_1 + \lambda_2{}^2(\frac{1}{E_2} - 1) \geq 0$$

Which is quadratic equation for ‘ λ_1 ’

Where $a = (\frac{1}{E_1} - 1), b = 2\lambda_2(\frac{\rho}{\sqrt{E_1, E_2}} - 1), c = \lambda_2{}^2(\frac{1}{E_2} - 1)$

The discriminate is

$b^2 - 4ac \geq 0 \therefore$ Because roots are real

$$[2\lambda_2(\frac{\rho}{\sqrt{E_1, E_2}} - 1)]^2 - 4[(\frac{1}{E_1} - 1)(\lambda_2{}^2(\frac{1}{E_2} - 1)] \geq 0$$

$$4\lambda_2^2(\frac{\rho}{\sqrt{E_1, E_2}} - 1)^2 - 4\lambda_2{}^2(\frac{1}{E_1} - 1)(\frac{1}{E_2} - 1) \geq 0$$

$$(\frac{\rho}{\sqrt{E_1,E_2}}-1)^2-(\frac{1}{E_1}-1)(\frac{1}{E_2}-1)\geq 0 \qquad \therefore\ 4\lambda^2{}_2\neq 0$$

$$[\frac{\rho^2}{E_1E_2}+1-2\frac{\rho}{\sqrt{E_1,E_2}}]-\frac{1}{E_1E_2}+\frac{1}{E_1}+\frac{1}{E_2}-1\geq 0$$

$$\frac{\rho^2}{E_1E_2}+1-2\frac{\rho}{\sqrt{E_1,E_2}}-\frac{1}{E_1E_2}+\frac{1}{E_1}+\frac{1}{E_2}-1\geq 0$$

$$\frac{\rho^2}{E_1E_2}-2\frac{\rho}{\sqrt{E_1,E_2}}+\frac{1}{E_1E_2}(-1+E_2+E_1)\geq 0$$

Multiplying by $E_1\ E_2$

$$\rho^2-2\sqrt{E_1,E_2}\,\rho+(-1+E_1+E_2)\geq 0$$

Which is also a quadratic equation for ρ

$$\rho^2-(2\sqrt{E_1,E_2})\rho+(E_1+E_2-1)\geq 0$$

$$a=1, b=(2\sqrt{E_1,E_2})\rho, c=(E_1+E_2-1)$$

$$\rho=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\rho=\frac{(2\sqrt{E_1,E_2})\pm\sqrt{4(E_1E_2)-4(E_1+E_2-1)(1)}}{2(1)}$$

$$\rho=\frac{(2\sqrt{E_1,E_2})\pm 2\sqrt{(E_1E_2)-(E_1+E_2-1)(1)}}{2}$$

$$\rho=\frac{2[(\sqrt{E_1,E_2})\pm\sqrt{(E_1E_2)-(E_1+E_2-1)(1)}]}{2}$$

$$\rho=[(\sqrt{E_1,E_2})\pm\sqrt{(E_1E_2)-(E_1+E_2-1)}]$$

$$\rho=(\sqrt{E_1,E_2})\pm\sqrt{E_1E_2-E_1-E_2+1)}$$

$$\rho=(\sqrt{E_1,E_2})\pm\sqrt{E_1(E_2-1)-1(E_2-1)}$$

$$\rho=(\sqrt{E_1,E_2})\pm\sqrt{(E_1-1)(E_2-1)}$$

$$\sqrt{E_1,E_2}-\sqrt{(E_1-1)(E_2-1)}\leq\rho\leq\sqrt{E_1,E_2}+\sqrt{(E_1-1)(E_2-1)}$$

$$\sqrt{E_1,E_2}-\sqrt{-1(1-E_1)-1(1-E_2)}\leq\rho\leq\sqrt{E_1,E_2}+\sqrt{-1(1-E_1)-1(1-E_2)}$$

$$\sqrt{E_1,E_2}-\sqrt{(1-E_1)(1-E_2)}\leq\rho\leq\sqrt{E_1,E_2}+\sqrt{(1-E_1)(1-E_2)}$$

As required result